# A Schur Lemma for Homotopy Representations

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**Abstract**: A p-group P acts “ irreducibly” on a mod-p homology sphere X if its dimension function is that of an irreducible real representation. In analogy with ordinary representation theory we prove here that if a cyclic group C, of prime order acts on X, commutes with the P action and its fixed set is a mod-p homology sphere then either that fixed set is empty or has the homology of X.

**Introduction**. If a p-group P acts on a mod-p homology sphere *X*, it is well-known that the fixed set of any subgroup H, $X^H$ has the mod-p homology of a sphere $S^{n(H)}$. If the fixed set should be empty, then we set *n(H)* to be -1. The function $H \to n(H)$ from the subgroups of P to the integers so-defined is called the “dimension function” and has a sizable literature ( see [ N ] and references there ). By [DH ] (or [tD2 ]) it is known that this function arises as the dimension function of a linear representation. That is, there exists a real representation V of P such that $X^H$ has the homology of $S(V)^H$ (where *S(V)* denotes the unit sphere of V). When V is in fact irreducible we will say that P “acts irreducibly” on X. By [LW] or [tD1 ], this is well-defined . If V is an irreducible representation of P the Schur Lemma states that if T is any equivariant linear transformation then the kernel of T is either 0 or all of V. When P acts irreducibly on X we will prove the following analogue:

**Theorem**: Let a p-group P act irreducibly on the mod-p homology n-sphere X and suppose that C is a cyclic group of prime order q which acts on X commuting with the action of P. Then if the fixed set of C is a mod-p homology m-sphere, this fixed set is either empty or has the homology of X.

Recall that a “homotopy representation” of a finite group G is a finite G-CW complex such that the fixed set of each subgroup has the homotopy type of a sphere.

**Corollary**: Suppose that X is a homotopy representation of the finite group G and that some Sylow subgroup P of G acts irreducibly on X. Then the fixed set of any cyclic subgroup of G lying in the centraliser of P is either empty or has the homology of X.

**Proof of Theorem**: We will make use of the notation in [DH]. We argue by induction on the order of P and on n. First we may as well assume that P acts faithfully otherwise we would have an irreducible action of a smaller order p-group. Since P has a faithful, irreducible representation it must have a cyclic center. If P is actually a cyclic group then n=1 (if $|P|$>2) or n=0 (if $|P|$=2 ). In these cases it is easy to see that if the fixed set of C is nonempty then m=n. If P is not cyclic then either P has a normal elementary abelian subgroup A of rank 2 or p = 2 and P is dihedral, semi-dihedral or generalised-quaternionic. First, assume that P has a normal elementary abelian subgroup A of rank 2. Then P acts by conjugation on A and determines a homomorphism $\phi : P \to Aut(A)$. If H is the kernel of $\phi$ then, as in [DH ], $V = Ind_H^P(\hat{V})$ is a representation of P which realises the dimension function for P and so must be irreducible. $\hat{V}$ is the representation of H obtained from the dimension function of the H action on $X^{\hat{\mathbb{Z}}_p}$ where $\hat{\mathbb{Z}}_p$ is any non P-central subgroup of A of order p. There are p of these and they are all conjugate in P. Since $Ind_H^P$ is additive, $\hat{V}$ must be irreducible too. If we denote the fixed set of C by Y, A acts on Y and $Y^{\hat{\mathbb{Z}}_p} = Y \cap X^{\hat{\mathbb{Z}}_p}$. If $Y^{\hat{\mathbb{Z}}_p}$ were empty then A would act freely on Y which is not possible. By induction, we must have $Y^{\hat{\mathbb{Z}}_p} = X^{\hat{\mathbb{Z}}_p}$. By the Borel formula we must have,

$$m+1 = p(n(\hat{\mathbb{Z}}_p)+1)$$

and similarly,

$$n+1 = p(n(\hat{\mathbb{Z}}_p)+1)$$

Then clearly, m=n.

On the other hand, if P does not have such a normal abelian subgroup A, then p = 2 and P is either generalised quaternion, dihedral or semi-dihedral ( see [G]). Since P acts irreducibly, no normal subgroup has fixed points on X and so no normal subgroup has fixed points on Y since Y is P invariant. As a result m = n = 4 in the case of P being generalised quaternion or semi-dihedral and m = n = 2 in the case of P being dihedral (as in [DH ]).

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