

# Smooth $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -actions on the $(2m + 2n - 1)$ -sphere

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## Abstract

We shall study smooth  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -actions on the natural  $(2m+2n-1)$ -sphere whose restricted action to the maximal compact subgroup is standard. We shall show that such an action is characterized by a smooth  $\mathbf{R}^2$ -action on the 3-sphere and consequently there exist uncountably many topologically distinct such actions.

## 1 Introduction.

Let  $\psi$  be the orthogonal  $SU(m) \times SU(n)$ -action on the natural  $(2m + 2n - 1)$ -sphere. If  $m, n \geq 2$ , then the action has codimension-one principal orbits with  $SU(m-1) \times SU(n-1)$  as the principal isotropy subgroup. The action restricted to the principal isotropy subgroup has the fixed point set  $F$  diffeomorphic to  $S^3$ .

In this paper, we shall study smooth  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -actions on the sphere for  $m, n \geq 4$  whose restricted action to the maximal compact subgroup  $SU(m) \times SU(n)$  is  $\psi$ .

The characterizaton of such actions of non-compact simple Lie groups on a standard sphere was first introduced by Asoh [1] and improved by [6], [3] and [7].  $SL(m, \mathbf{R}) \times SL(n, \mathbf{R})$ -actions and  $SL(m, \mathbf{H}) \times SL(n, \mathbf{H})$ -actions are studied by [8] and [9], respectively.

In §2, we shall give some subgroups of  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$  using in the later. In §3, we shall give examples of smooth  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -actions on the  $(2m + 2n - 1)$ -sphere with one open orbit. The other sections, we shall construct such actions on the sphere with plural open orbits.

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## 2 Certain subgroups of $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ .

Let  $L(n), L^*(n), N(n)$  and  $N^*(n)$  denote the closed connected subgroups of  $SL(n, \mathbf{C})$  consisting of matrices in the form

$$\left[ \begin{array}{c|ccc} 1 & * & \dots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right], \left[ \begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline * & & & \\ \vdots & & * & \\ * & & & \end{array} \right], \left[ \begin{array}{c|ccc} * & * & \dots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right] \text{ and } \left[ \begin{array}{c|ccc} * & 0 & \dots & 0 \\ \hline * & & & \\ \vdots & & * & \\ * & & & \end{array} \right],$$

respectively.

Let  $\sigma$  be the automorphism of  $SL(n, \mathbf{C})$  defined by  $\sigma(A) = A^{*-1}$ . Then the restriction  $\sigma|_{SU(n)}$  is the identity on  $SU(n)$  and we see  $\sigma(L^*(n)) = L(n)$  and  $\sigma(N^*(n)) = N(n)$ . We have the following lemma. The proof is same as [9]. Hence we omit it.

**LEMMA 2.1.** *Suppose  $m \geq 4$  and  $n \geq 4$ . Let  $S$  be a closed subgroup of  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$  such that the natural  $SU(m) \times SU(n)$ -action on the homogeneous space  $(SL(m, \mathbf{C}) \times SL(n, \mathbf{C}))/S$  has at most three isotropy types  $SU(m) \times SU(n-1), SU(m-1) \times SU(n)$  and  $SU(m-1) \times SU(n-1)$ . Then*

(1) *If  $S$  contains  $SU(m) \times SU(n-1)$ , then  $SL(m, \mathbf{C}) \times L(n) \subset S \subset SL(m, \mathbf{C}) \times N(n)$ .*

(2) *If  $S$  contains  $SU(m-1) \times SU(n)$ , then  $L(m) \times SL(n, \mathbf{C}) \subset S \subset N(m) \times SL(n, \mathbf{C})$ .*

(3) *If  $S$  contains  $SU(m-1) \times SU(n-1)$  but  $S$  does not contain  $SU(m) \times SU(n-1)$  nor  $SU(m-1) \times SU(n)$ , then  $L(m) \times L(n) \subset S \subset N(m) \times N(n)$ .*

*Here the results are described up to an equivalence under automorphisms of  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$  which leave invariant the subgroup  $SU(m) \times SU(n)$ .*

Now we also define the closed connected subgroups  $N_n(a)$  of  $N(n)$  and  $N(a : b)$  of  $N(m) \times N(n)$  consisting of the matrices in the form

$$\left[ \begin{array}{c|ccc} \exp a\theta & * & \dots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right], \left( \left[ \begin{array}{c|ccc} \exp a\theta & * & \dots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right], \left[ \begin{array}{c|ccc} \exp b\theta & * & \dots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right] \right),$$

respectively. Here  $\theta \in \mathbf{R}$  and  $a, b$  are complex numbers satisfying  $(a, b) \neq (0, 0)$ .

## 3 Twisted linear actions.

Let  $\alpha, \beta$  be complex numbers satisfying  $\operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0$ . Then there is the smooth action  $\Phi_{\alpha, \beta}$  of  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$  on  $S^{2m+2n-1}$  defined by

$$\Phi_{\alpha, \beta}((A, B), \mathbf{u} \oplus \mathbf{v}) = \exp(\alpha\theta)A\mathbf{u} \oplus \exp(\beta\theta)B\mathbf{v}, \quad (3.1)$$

where  $(A, B) \in SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ ,  $\mathbf{u} \oplus \mathbf{v} \in S^{2m+2n-1} \subset \mathbf{C}^m \oplus \mathbf{C}^n$  and  $\theta$  is a certain real number. The restricted  $SU(m) \times SU(n)$ -action is the standard action  $\psi$ . The action  $\Phi_{\alpha, \beta}$  is called the twisted linear action. We see the action  $\Phi_{\alpha, \beta}$  has just three orbits; One is an open orbit and the others are compact orbits. Especially, if  $\text{Re } \alpha = \text{Re } \beta$ , then the righthand side of (3.1) is presented as follows.

$$\exp(-(1 + ic_1) \log \|A\mathbf{u} \oplus B\mathbf{v}\|)A\mathbf{u} \oplus \exp(-(1 + ic_2) \log \|A\mathbf{u} \oplus B\mathbf{v}\|)B\mathbf{v},$$

where  $c_1, c_2$  are certain real numbers. We denote the action by  $\Phi_{(c_1, c_2)}$ . The isotropy types of this action are  $N(1 + ic_1 : 1 + ic_2)$  at the open orbit and  $N_m(1 + ic_1) \times SL(n, \mathbf{C})$  and  $SL(m, \mathbf{C}) \times N_n(1 + ic_2)$  at the compact orbits.

We say two actions  $\Phi_{(c_1, c_2)}, \Phi_{(c'_1, c'_2)}$  of  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$  on  $S^{2m+2n-1}$  are equivalent if there exists an  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -equivariant diffeomorphism.

Considering the restriction of the equivariant diffeomorphism to the two compact orbits, we have the following.

**LEMMA 3.1.** *If two actions  $\Phi_{(c_1, c_2)}, \Phi_{(c'_1, c'_2)}$  are equivalent, then  $(c_1, c_2) = (c'_1, c'_2)$ .*

**COROLLARY 3.2.** *There exist non-countably many non-equivalent smooth  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -actions on  $S^{2m+2n-1}$  with one open orbit whose restricted  $SU(m) \times SU(n)$ -action is  $\psi$ .*

## 4 Smooth $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -actions on $S^{2m+2n-1}$

We consider smooth  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -actions on  $S^{2m+2n-1}$  whose restricted  $SU(m) \times SU(n)$ -action is  $\psi$ . The action  $\psi$  has codimension-one principal orbits with  $SU(m-1) \times SU(n-1)$  as the principal isotropy subgroup and has two singular orbits. We put  $F = \{u\mathbf{e}_1 + v\mathbf{e}_{m+1} \mid |u|^2 + |v|^2 = 1\}$ , where  $\mathbf{e}_1, \dots, \mathbf{e}_m, \mathbf{e}_{m+1}, \dots, \mathbf{e}_{m+n}$  is the orthonormal standard basis of  $\mathbf{C}^m \oplus \mathbf{C}^n$ . Then  $F$  is the fixed points set of the restricted  $SU(m-1) \times SU(n-1)$ -action of  $\psi$  on  $S^{2m+2n-1}$ . Hereafter we define

$$\begin{aligned} G &= SL(m, \mathbf{C}) \times SL(n, \mathbf{C}), \\ K &= SU(m) \times SU(n), \\ H &= SU(m-1) \times SU(n-1). \end{aligned}$$

Let  $\Phi : G \times S^{2m+2n-1} \rightarrow S^{2m+2n-1}$  be a smooth  $G$ -action whose restricted  $K$ -action is  $\psi$ . We denote by  $F(S)$  the fixed point set of the restricted  $S$ -action of the action  $\Phi$  on  $S^{2m+2n-1}$ . Then we see that  $F = F(H)$ .

**LEMMA 4.1.** *For any smooth  $G$ -action on  $S^{2m+2n-1}$  whose restricted  $K$ -action is  $\psi$ , the set  $F$  coincides with  $F(L(m) \times L(n)), F(L^*(m) \times L(n)), F(L(m) \times L^*(n))$  or  $F(L^*(m) \times L^*(n))$ .*

The proof is same as [8],[9]. By this lemma and the automorphism  $\sigma$  of  $G$ , we may assume  $F = F(L(m) \times L(n))$  for the action  $\Phi$ . Moreover, we have the following.

LEMMA 4.2. Let  $\Phi : G \times S^{2m+2n-1} \rightarrow S^{2m+2n-1}$  be a smooth  $G$ -action whose restricted  $K$ -action is  $\psi$  and  $F = F(L(m) \times L(n))$ . Suppose  $x \in F$ . Then

(1) If  $G_x$  contains  $SU(m-1) \times SU(n)$ , then  $G_x = N_n(1+ic_1) \times SL(m, \mathbf{C})$  for some  $c_1 \in \mathbf{R}$

(2) If  $G_x$  contains  $SU(m) \times SU(n-1)$ , then  $G_x = SL(m, \mathbf{C}) \times N_n(1+ic_2)$  for some  $c_2 \in \mathbf{R}$

*Proof.* We shall show the case (2). By Lemma 2.1, we have

$$SL(m, \mathbf{C}) \times L(n) \subset G_x \subset SL(m, \mathbf{C}) \times N(n).$$

By the dimension of the orbit, we have  $\dim G_x \neq \dim (SL(m, \mathbf{C}) \times N(n))$ . Suppose  $\dim G_x = \dim (SL(m, \mathbf{C}) \times L(n))$ . Then the orbit  $G(x)$  is  $2n$  dimensional and must contain a principal orbit of the restricted  $K$ -action  $\psi$ . This is a contradiction. Hence we obtain the result.

We denote the elements  $D_{1,c_1}(\theta)$ ,  $D_{2,c_2}(\theta)$ ,  $D_1(t)$  and  $D_2(t)$  by

$$D_{1,c_1}(\theta) = \left[ \begin{array}{c|ccc} \exp(1+ic_1)\theta & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & xI_{m-1} & \end{array} \right] \in C_{SL(m, \mathbf{C})}(SU(m-1)),$$

$$D_{2,c_2}(\theta) = \left[ \begin{array}{c|ccc} \exp(1+ic_2)\theta & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & & yI_{n-1} & \end{array} \right] \in C_{SL(n, \mathbf{C})}(SU(n-1)),$$

$$D_1(t) = \text{diag}(t, t^{-1}, 1, \dots, 1) \in N_{SU(m)}(SU(m-1)),$$

$$D_2(t) = \text{diag}(t, t^{-1}, 1, \dots, 1) \in N_{SU(n)}(SU(n-1)),$$

respectively, where  $\theta, c_1, c_2 \in \mathbf{R}, t \in U(1), x^{m-1} \exp(1+ic_1)\theta = 1, y^{n-1} \exp(1+ic_2)\theta = 1, C_A(B)$  is the centralizer of  $B$  in  $A$  and  $N_A(B)$  is the normalizer of  $B$  in  $A$ . We put

$$D_{1,c_1}(\mathbf{R}) = \{D_{1,c_1}(\theta) | \theta \in \mathbf{R}\},$$

$$D_{2,c_2}(\mathbf{R}) = \{D_{2,c_2}(\theta) | \theta \in \mathbf{R}\},$$

$$M(c_1, c_2) = D_{1,c_1}(\mathbf{R}) \times D_{2,c_2}(\mathbf{R})$$

$$T = \{(D_1(t_1), D_2(t_2)) | t_1, t_2 \in U(1)\}.$$

Now we have the following decompositions by the slight deformation of the Iwasawa decompositions of the Lie groups  $SL(m, \mathbf{C})$  and  $SL(n, \mathbf{C})$ .

$$SL(m, \mathbf{C}) = SU(m)D_{1,c_1}(\mathbf{R})L(m), SL(n, \mathbf{C}) = SU(n)D_{2,c_2}(\mathbf{R})L(n), \quad (4.1)$$

where  $c_1, c_2$  are any real numbers.

Let  $\Phi : G \times S^{2m+2n-1} \longrightarrow S^{2m+2n-1}$  be a smooth  $G$ -action whose restricted  $K$ -action is  $\psi$  and  $F = F(L(m) \times L(n))$ . Then by Lemma 4.2 we have

$$G_{\mathbf{e}_1} \supset N_m(1 + ic_1), G_{\mathbf{e}_{m+1}} \supset N_n(1 + ic_2)$$

for some real numbers  $c_1, c_2$ . The restricted  $M(c_1, c_2)$ -action on  $F$  implies the  $\mathbf{R}^2$ -action  $\Phi_{M(c_1, c_2)} : \mathbf{R}^2 \times F \longrightarrow F$  defined by

$$\Phi_{M(c_1, c_2)}((\theta_1, \theta_2), (u, v)) = \Phi((D_{1, c_1}(\theta_1), D_{2, c_2}(\theta_2)), (u, v)),$$

where  $(u, v) = u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1} \in F$ . We can describe

$$\Phi_{M(c_1, c_2)}((\theta_1, \theta_2), (u, v)) = (a(\theta_1, \theta_2, u, v), b(\theta_1, \theta_2, u, v))$$

for certain complex valued functions  $a, b$ . Since  $D_{s, c_s}(\theta_s)D_s(t_s) = D_s(t_s)D_{s, c_s}(\theta_s)$  ( $s = 1, 2$ ), we have

$$\begin{aligned} a(\theta_1, \theta_2, t_1u, t_2v) &= t_1a(\theta_1, \theta_2, u, v), \\ b(\theta_1, \theta_2, t_1u, t_2v) &= t_2b(\theta_1, \theta_2, u, v), \end{aligned}$$

for any  $(t_1, t_2) \in T$ . Hence we see that there exist smooth complex functions  $\alpha, \beta$  on  $\mathbf{R}^2 \times F$  satisfying

$$\begin{aligned} a(\theta_1, \theta_2, u, v) &= u\alpha(\theta_1, \theta_2, u, v), \\ b(\theta_1, \theta_2, u, v) &= v\beta(\theta_1, \theta_2, u, v). \end{aligned}$$

the functions  $\alpha, \beta$  satisfy the following relations:

$$\begin{aligned} \alpha(\theta_1, \theta_2, t_1u, t_2v) &= \alpha(\theta_1, \theta_2, u, v), \\ \beta(\theta_1, \theta_2, t_1u, t_2v) &= \beta(\theta_1, \theta_2, u, v), \end{aligned}$$

for any  $(t_1, t_2) \in T$ . We also have

$$\alpha(0, 0, u, v) = \beta(0, 0, u, v) = 1.$$

**THEOREM 4.3.** *Let  $\Phi, \Phi' : G \times S^{2m+2n-1} \longrightarrow S^{2m+2n-1}$  be smooth  $G$ -actions whose restricted  $K$ -action is  $\psi$  and  $F = F(L(m) \times L(n))$ . Suppose  $\Phi_{M(c_1, c_2)} = \Phi'_{M(c'_1, c'_2)}$  and  $(c_1, c_2) = (c'_1, c'_2)$ . Then  $\Phi = \Phi'$*

*Proof.* For the action  $\Phi$  we have real numbers  $c_1, c_2$  satisfying

$$G_{\mathbf{e}_1} \supset N_m(1 + ic_1), G_{\mathbf{e}_{m+1}} \supset N_n(1 + ic_2).$$

Using the numbers  $c_1, c_2$ , we decompose  $SL(m, \mathbf{C}), SL(n, \mathbf{C})$  as (4.1).

Let  $g = (g_1, g_2) \in G$  and decompose  $g_s$  as

$$g_s = k_s D_{s, c_s}(\theta_s) l_s \quad (s = 1, 2).$$

Then we have

$$\Phi((g_1, g_2), u\mathbf{e}_1 \oplus v\mathbf{e}_{m+1}) = \alpha u k_1 \mathbf{e}_1 \oplus \beta v k_2 \mathbf{e}_{m+1}.$$

We also have

$$\theta_1 = \log \frac{\|g_1 u e_1\|}{\|u e_1\|}, \quad \theta_2 = \log \frac{\|g_2 v e_{m+1}\|}{\|v e_{m+1}\|},$$

for  $uv \neq 0$ . Hence we obtain

$$\begin{aligned} & \Phi((g_1, g_2), u e_1 \oplus v e_{m+1}) \\ &= \alpha \exp(-(1 + ic_1) \log \frac{\|g_1 u e_1\|}{\|u e_1\|}) g_1 u e_1 \oplus \beta \exp(-(1 + ic_2) \log \frac{\|g_2 v e_{m+1}\|}{\|v e_{m+1}\|}) g_2 v e_{m+1} \end{aligned}$$

for  $u > 0, v > 0$ . Here

$$\begin{aligned} \alpha &= \alpha(\log \frac{\|g_1 u e_1\|}{\|u e_1\|}, \log \frac{\|g_2 v e_{m+1}\|}{\|v e_{m+1}\|}, \|u e_1\|, \|v e_{m+1}\|), \\ \beta &= \beta(\log \frac{\|g_1 u e_1\|}{\|u e_1\|}, \log \frac{\|g_2 v e_{m+1}\|}{\|v e_{m+1}\|}, \|u e_1\|, \|v e_{m+1}\|). \end{aligned}$$

For any  $\mathbf{u} \oplus \mathbf{v} \in S^{2m+2n-1}$  satisfying  $\|\mathbf{u}\| \neq 0$  and  $\|\mathbf{v}\| \neq 0$ , there exists  $(k'_1, k'_2) \in K$  and  $u > 0, v > 0$  such that  $\mathbf{u} = k'_1 u e_1$  and  $\mathbf{v} = k'_2 v e_{m+1}$ . Since  $\Phi((g_1, g_2), \mathbf{u} \oplus \mathbf{v}) = \Phi((g_1 k'_1, g_2 k'_2), u e_1 \oplus v e_{m+1})$ , we obtain

$$\begin{aligned} & \Phi((g_1, g_2), \mathbf{u} \oplus \mathbf{v}) \\ &= \alpha \exp\left(- (1 + ic_1) \log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}\right) g_1 \mathbf{u} \oplus \beta \exp\left(- (1 + ic_2) \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}\right) g_2 \mathbf{v}, \end{aligned}$$

where

$$\begin{aligned} \alpha &= \alpha(\log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}, \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}, \|\mathbf{u}\|, \|\mathbf{v}\|), \\ \beta &= \beta(\log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}, \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}, \|\mathbf{u}\|, \|\mathbf{v}\|). \end{aligned}$$

For  $\|\mathbf{u}\| \|\mathbf{v}\| = 0$ , we see

$$\begin{aligned} \Phi((g_1, g_2), \mathbf{u} \oplus \mathbf{0}) &= \exp(-(1 + ic_1) \log \|g_1 \mathbf{u}\|) g_1 \mathbf{u} \oplus \mathbf{0}, \\ \Phi((g_1, g_2), \mathbf{0} \oplus \mathbf{v}) &= \mathbf{0} \oplus \exp(-(1 + ic_2) \log \|g_2 \mathbf{v}\|) g_2 \mathbf{v}. \end{aligned}$$

Hence we see that  $\Phi = \Phi'$  if  $\Phi_{M(c_1, c_2)} = \Phi'_{M(c'_1, c'_2)}$  and  $(c_1, c_2) = (c'_1, c'_2)$ .

## 5 Induced smooth $\mathbf{R}^2$ -action on $F$

Let  $\Phi : G \times S^{2m+2n-1} \longrightarrow S^{2m+2n-1}$  be a smooth  $G$ -action whose restricted  $K$ -action is  $\psi$  and  $F = F(L(m) \times L(n))$  and let  $\Phi_{M(c_1, c_2)}$  be the induced  $\mathbf{R}^2$ -action on  $F$ . We denote  $X = (u, v) \in F$  and  $\Theta = (\theta_1, \theta_2) \in \mathbf{R}^2$ . Then the action  $\Phi_{(c_1, c_2)}$  is described by

$$\Phi_{M(c_1, c_2)}(\Theta, X) = (u\alpha(\Theta, X), v\beta(\Theta, X))$$

for certain complex valued smooth functions  $\alpha, \beta$  on  $\mathbf{R}^2 \times F$  satisfying the equations

- (i)  $\alpha(\Theta, \psi((t_1, t_2), X)) = \alpha(\Theta, X)$ ,  $\beta(\Theta, \psi((t_1, t_2), X)) = \beta(\Theta, X)$ ,
- (ii)  $\alpha(O, X) = \beta(O, X) = 1$ ,

where  $(t_1, t_2) \in T$  (see §4). Since  $\Phi_{M(c_1, c_2)}$  is the  $\mathbf{R}^2$ -action, we also obtain the equations

- (iii)  $\alpha(\Theta' + \Theta, X) = \alpha(\Theta', (\alpha(\Theta, X)u, \beta(\Theta, X)v))\alpha(\Theta, X)$ ,
- (iv)  $\beta(\Theta' + \Theta, X) = \beta(\Theta', (\alpha(\Theta, X)u, \beta(\Theta, X)v))\beta(\Theta, X)$ .

By these equations, we have

$$\alpha(\Theta, X) \neq 0, \beta(\Theta, X) \neq 0$$

for any  $(\Theta, X) \in \mathbf{R}^2 \times F$ .

Let  $\alpha, \beta$  be complex valued smooth functions on  $\mathbf{R}^2 \times F$  satisfying the four conditions above and  $c_1, c_2$  be any real numbers. We define a mapping  $\tilde{\Phi} : G \times S^{2m+2n-1} \longrightarrow S^{2m+2n-1}$  by

$$\begin{aligned} & \tilde{\Phi}((g_1, g_2), \mathbf{u} \oplus \mathbf{v}) \\ &= \alpha \exp(-(1 + ic_1) \log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}) g_1 \mathbf{u} \oplus \beta \exp(-(1 + ic_2) \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}) g_2 \mathbf{v} \end{aligned}$$

where

$$\begin{aligned} \alpha &= \alpha(\log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}, \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}, \|\mathbf{u}\|, \|\mathbf{v}\|), \\ \beta &= \beta(\log \frac{\|g_1 \mathbf{u}\|}{\|\mathbf{u}\|}, \log \frac{\|g_2 \mathbf{v}\|}{\|\mathbf{v}\|}, \|\mathbf{u}\|, \|\mathbf{v}\|) \end{aligned}$$

for any  $\mathbf{u} \oplus \mathbf{v} \in S^{2m+2n-1}$  satisfying  $\|\mathbf{u}\| \neq 0$  and  $\|\mathbf{v}\| \neq 0$  and

$$\begin{aligned} \tilde{\Phi}((g_1, g_2), \mathbf{u} \oplus \mathbf{0}) &= \exp(-(1 + ic_1) \log \|g_1 \mathbf{u}\|) g_1 \mathbf{u} \oplus \mathbf{0}, \\ \tilde{\Phi}((g_1, g_2), \mathbf{0} \oplus \mathbf{v}) &= \mathbf{0} \oplus \exp(-(1 + ic_2) \log \|g_2 \mathbf{v}\|) g_2 \mathbf{v}. \end{aligned}$$

By a routine work, we have

**PROPOSITION 5.1.**  $\tilde{\Phi}$  is an abstract  $G$ -action on  $S^{2m+2n-1}$ .

However this action is not smooth in general.

**PROPOSITION 5.2.** If  $\alpha = \beta = 1$ , then  $\tilde{\Phi}$  is not smooth.

The proof is same as [8].

In the rest of this paper, we construct some smooth  $G$ -actions on  $S^{2m+2n-1}$ .

## 6 Construction of smooth $G$ -actions on $S^{2m+2n-1}$

### 6.1 Certain smooth actions of $SL(m, \mathbf{C})$ on $\mathbf{C}^m$

For any real number  $c$ , we define the map  $\varphi_{m,c} : SL(m, \mathbf{C}) \times \mathbf{C}^m \longrightarrow \mathbf{C}^m$  by

$$\varphi_{m,c}(A, \mathbf{u}) = s \exp(ic \log s) A \mathbf{u},$$

where

$$s = \frac{2}{1 - \|\mathbf{u}\|^2 + (4\|A\mathbf{u}\|^2 + (1 - \|\mathbf{u}\|^2)^2)^{\frac{1}{2}}}.$$

Then we see that  $\varphi_{m,c}$  is a smooth  $SL(m, \mathbf{C})$ -action on  $\mathbf{C}^m$  and the three sets

$$\{\mathbf{u} \in \mathbf{C}^m \mid \|\mathbf{u}\| < 1\}, \{\mathbf{u} \in \mathbf{C}^m \mid \|\mathbf{u}\| = 1\}, \{\mathbf{u} \in \mathbf{C}^m \mid \|\mathbf{u}\| > 1\}$$

are invariant under the action  $\varphi_{m,c}$ . Moreover, if  $\|\mathbf{u}\| \neq 1$ , then we have

$$\frac{\exp(-ic \log s)}{1 - \|\varphi_{m,c}(A, \mathbf{u})\|^2} \varphi_{m,c}(A, \mathbf{u}) = \frac{1}{1 - \|\mathbf{u}\|^2} A\mathbf{u}$$

and if  $\|\mathbf{u}\| = 1$ , then we have

$$\varphi_{m,c}(A, \mathbf{u}) = \exp(-(1 + ic) \log \|A\mathbf{u}\|) A\mathbf{u}.$$

We put  $D^{2m} = \{\mathbf{u} \in \mathbf{C}^m \mid \|\mathbf{u}\| < 1\}$  and define the map  $h : D^{2m} \rightarrow D^{2m}$  by

$$h(\mathbf{u}) = \frac{\exp(-ic \log(1 - \|\mathbf{u}\|^2))}{1 - \|\mathbf{u}\|^2} \mathbf{u}.$$

Then the map  $h$  is an equivariant diffeomorphism of  $D^{2m}$  with the action  $\varphi_{m,c}$  onto  $\mathbf{C}^m$  with the linear action. Now we define a smooth  $\mathbf{R}$ -action  $\phi_{m,c} : \mathbf{R} \times \mathbf{C} \rightarrow \mathbf{C}$  by

$$\phi_{m,c}(\theta, z)\mathbf{e}_1 = \varphi_{m,c}(D_{1,c}(\theta), z\mathbf{e}_1).$$

We see that the  $\mathbf{R}$ -action  $\phi_{m,c}$  on  $\mathbf{C}$  corresponds to the following vector field  $\eta_1$  on  $\mathbf{C}$ :

$$\eta_1 = \frac{1 - |z|^2}{1 + |z|^2} \left( (1 + ic)z \frac{d}{dz} + (1 - ic)\bar{z} \frac{d}{d\bar{z}} \right),$$

where we identify  $\mathbf{C}$  with  $\mathbf{R}^2$  by  $z = x + iy$  and put

$$\frac{d}{dz} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{d}{d\bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

## 6.2 Construction of smooth $G$ -actions on $S^{2m+2n-1}$

First, we shall construct a smooth one-parameter group on  $F$ .

Let  $k, l$  be diffeomorphisms of  $\mathbf{C} \times S^1$  onto open sets of  $F$  defined by

$$k(z, t) = \left( \frac{z}{\sqrt{16 + |z|^2}}, \frac{4t}{\sqrt{16 + |z|^2}} \right),$$

$$l(z, t) = \left( \frac{4t}{\sqrt{16 + |z|^2}}, \frac{z}{\sqrt{16 + |z|^2}} \right).$$

Let  $\rho(x)$  be a smooth real valued even function such that

$$\begin{aligned}\rho(x) &= 1 \text{ for } |x| \leq 1, \\ \rho(x) &= 0 \text{ for } |x| \geq 2, \\ 0 &< \rho(x) < 1 \text{ for } 1 < |x| < 2\end{aligned}$$

and monotone decreasing at  $1 < x < 2$  and let  $\sigma(z)$  be the smooth function on  $\mathbf{C}$  defined by  $\sigma(z) = \rho(|z|)$ .

Let  $\eta_2$  be the tangent vector field induced from the  $\mathbf{R}^2$ -action  $\Phi_{M(0,0)}$  on  $F$ . Namely, we put

$$\eta_2 = |u|^2 \left( v \frac{\partial}{\partial v} + \bar{v} \frac{\partial}{\partial \bar{v}} \right) - |v|^2 \left( u \frac{\partial}{\partial u} + \bar{u} \frac{\partial}{\partial \bar{u}} \right).$$

Define a tangent vector field  $\xi$  on  $F$  by

$$\xi = k_*(\sigma(z^2)\eta_1 + O) + l_*(\sigma(z^2)\eta_1 + O) + \left(\sigma\left(\frac{12}{5}u^2\right) - \sigma(6u^2)\right)\eta_2,$$

where  $O$  is the zero vector field on  $S^1$ .

Let  $\phi$  be the one-parameter group on  $F$  corresponding to the vector field  $\xi$ . Since the vector field  $\xi$  is invariant under the standard  $T$ -action on  $F$ , we see that

$$\psi((D_1(t_1), D_2(t_2)), \phi(\theta, (u, v))) = \phi(\theta, \psi((D_1(t_1), D_2(t_2)), (u, v))) \quad (6.1)$$

for  $(t_1, t_2) \in T$ .

Now we define a smooth  $\mathbf{R}^2$ -action  $\varphi$  on  $F$  by

$$\varphi((\theta_1, \theta_2), (u, v)) = \begin{cases} \phi(\theta_1, (u, v)) & \text{for } |u|^2 < \frac{1}{9}, \\ \phi(\theta_2, (u, v)) & \text{for } |u|^2 > \frac{8}{9} \\ \phi(a\theta_1 + b\theta_2, (u, v)) & \text{for } \frac{1}{9} < |u|^2 < \frac{8}{9}, \end{cases}$$

where  $a, b$  are real numbers. Since  $\xi = 0$  for  $\frac{1}{9} < |u|^2 < \frac{1}{6}$  and  $\frac{5}{6} < |u|^2 < \frac{8}{9}$ , we see that  $\varphi$  is smooth on  $F$ .

By the use of (6.1), we can put  $\varphi((\theta_1, \theta_2), (u, v)) = (u\alpha(\theta_1, \theta_2, u, v), v\beta(\theta_1, \theta_2, u, v))$  for some complex valued functions  $\alpha, \beta$  on  $\mathbf{R}^2 \times F$ , where the functions satisfy four conditions in §4. Thus we can construct an abstract  $G$ -action  $\tilde{\Phi}$  on  $S^{2m+2n-1}$  defined in §4

Now we show that  $\tilde{\Phi}$  is smooth on neighborhoods of  $\mathbf{u} = \mathbf{0}$  and  $\mathbf{v} = \mathbf{0}$ . Let  $\tilde{k} : \mathbf{C}^m \times S^{2n-1} \rightarrow S^{2m+2n-1}$  be the diffeomorphism onto an open set of  $S^{2m+2n-1}$  defined by

$$\tilde{k}(\mathbf{u}, \mathbf{v}) = \frac{1}{\sqrt{16 + \|\mathbf{u}\|^2}} \mathbf{u} \oplus \frac{4}{\sqrt{16 + \|\mathbf{u}\|^2}} \mathbf{v}.$$

Then we see

$$\tilde{\Phi}((A, B), \mathbf{u} \oplus \mathbf{v}) = \tilde{k} \left( \varphi_{m,c}(A, \frac{4}{\sqrt{16 + \|\mathbf{u}\|^2}} \mathbf{u}), \frac{1}{\|B\mathbf{v}\|} B\mathbf{v} \right)$$

for  $\|\mathbf{u}\|^2 < \frac{1}{17}$ . Hence  $\tilde{\Phi}$  is smooth on a neighborhood of  $\mathbf{u} = \mathbf{0}$ . Similarly, we may show that the smoothness of  $\tilde{\Phi}$  on a neighborhood of  $\mathbf{v} = \mathbf{0}$ .

Therefore we see that the  $G$ -action  $\tilde{\Phi}$  on  $S^{2m+2n-1}$  is smooth. By this  $G$ -action, we obtain the following:

**THEOREM 6.1.** *There exists a smooth  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -action on  $S^{2m+2n-1}$  such that it has five open orbits and the isotropy types of them are the following :  $L(m) \times L_n(1)$  (two orbits) ,  $N((1+ic)b : -a)$ ,  $N_m(1) \times L(n)$  (two orbits) for any real numbers  $a, b, c$  such that  $(a, b) \neq (0, 0)$  .*

**COROLLARY 6.2.** *There are uncountably many topologically distinct smooth  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -actions on  $S^{2m+2n-1}$  with five open orbits whose restrict  $SU(m) \times SU(n)$ -action is  $\psi$  .*

**Remark 6.3.** *By an exchange of the third term of the vector field  $\xi$ , we may obtain another smooth  $SL(m, \mathbf{C}) \times SL(n, \mathbf{C})$ -action on  $S^{2m+2n-1}$  with  $(4k+1)$  open orbits.*

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