COMPLETION AND DIFFERENTIABILITY IN WEAKLY O-MINIMAL STRUCTURES

HIROSHI TANAKA AND TOMOHIRO KAWAKAMI

Abstract. Let $\mathcal{R} = (R, <, +, \cdot, \dots)$ be a non-valuational weakly o-minimal real closed field, I a definable convex open subset of R and $f : I \to \overline{R}$ a definable function. We prove that $\{x \in I : f'(x) \text{ exists in } \overline{R}\}$ is definable and f' is definable if f is differentiable.

1. Introduction

Weak o-minimality was introduced by M. A. Dickmann (see [2]). After that several fundamental results of weakly o-minimality were proved by D. Macpherson, D. Marker and C. Steinhorn in [3].

Non-valuational weakly o-minimal expansions of ordered groups and ordered fields were studied by D. Macpherson, D. Marker and C. Steinhorn in [3], by R. Wencel in [6]. Now, it is known that the model theory of weakly o-minimal structures does not develop as smoothly as that of o-minimal structures, see [3]. However non-valuational weakly o-minimal expansions of ordered groups are very similar to o-minimal structures. In particular, R. Wencel showed that every non-valuational weakly o-minimal expansion of an ordered group admits an o-minimal style cell decomposition in [6].

On the other hand, differentiability and analyticity properties of definable functions for weakly o-minimal expansions of real closed fields are scarcely studied (see [3, Open problem 3]).

Throughout this paper, "definable" means "definable possibly with parameters" and we assume that the structure $\mathcal{R} = (R, <, \dots)$ is an expansion of $(R, <)$ equipped with a dense linear ordering $<$ without endpoints. The set of positive integers is denoted by $\mathbb{N}$.

2. Preliminaries

A subset A of R is said to be *convex* if $a, b \in A$ and $c \in R$ with $a < c < b$ then $c \in A$. Moreover if $A = \emptyset$ or $\inf A, \sup A \in R \cup \{-\infty, +\infty\}$,

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then A is called an *interval* in R. We say that $\mathcal{R}$ is *o-minimal* (*weakly o-minimal*) if every definable subset of R is a finite union of intervals (convex sets), respectively.

For any subsets C, D of R, we write $C < D$ if $c < d$ whenever $c \in C$ and $d \in D$. A pair $\langle C, D\rangle$ of non-empty subsets of R is called a *cut* in $\mathcal{R}$ if $C < D, C \cup D = R$ and D has no lowest element. A cut $\langle C, D\rangle$ is said to be *definable* in $\mathcal{R}$ if the sets C, D are definable in $\mathcal{R}$. The set of all cuts definable in $\mathcal{R}$ will be denoted by $\overline{R}$. Note that we have $R = \overline{R}$ if $\mathcal{R}$ is o-minimal. We define a linear ordering on $\overline{R}$ by $\langle C_1, D_1\rangle < \langle C_2, D_2\rangle$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(R, <)$ as a substructure of $(\overline{R}, <)$ by identifying an element $a \in R$ with the definable cut $\langle(-\infty, a], (a, +\infty)\rangle$. We equip R ($\overline{R}$) with the *interval topology* (the open intervals form a base), and each product R^n ($(\overline{R})^n$) with the corresponding product topology, respectively.

Recall the notion of definable functions from [6]. Let $n \in \mathbb{N}$ and $A \subseteq R^n$ definable. A function $f : A \to \overline{R}$ is said to be *definable* if the set $\Gamma_<(f) := \{\langle x, y\rangle \in R^{n+1} : x \in A, y < f(x)\}$ is definable. A function $f : A \to \overline{R} \cup \{-\infty, +\infty\}$ is said to be *definable* if f is a definable function from A to $\overline{R}$, $f(x) = -\infty$ for all $x \in A$, or $f(x) = +\infty$ for all $x \in A$.

Let $\mathcal{R} = (R, <, +, \dots)$ be a weakly o-minimal expansion of an ordered group $(R, <, +)$. By [3, Theorem 5.1], the structure $\mathcal{R}$ is divisible and abelian. A cut $\langle C, D\rangle$ is said to be *non-valuational* if $\inf\{y - x : x \in C, y \in D\} = 0$. We say that the structure $\mathcal{R}$ is *non-valuational* if all cuts definable in $\mathcal{R}$ are non-valuational.

Let $\mathcal{R} = (R, <, +, \cdot, \dots)$ be a non-valuational weakly o-minimal expansion of an ordered field $(R, <, +, \cdot)$. By [3, Theorem 5.3], the structure $\mathcal{R}$ is real closed. For any subsets A, B of R, we define $A + B := \{x + y : x \in A, y \in B\}$, $A - B := \{x - y : x \in A, y \in B\}$, $A \cdot B := \{x \cdot y : x \in A, y \in B\}$ and $-A := \{-x : x \in A\}$. Note that $C - D = (-\infty, 0)$ and $D - C = (0, +\infty)$ for any element $\langle C, D\rangle$ of $\overline{R}$. Also notice that for any positive element $\langle C, D\rangle$ of $\overline{R}$ we have $\inf\{x^{-1} \cdot y : x \in C \cap (0, +\infty), y \in D\} = 1$, and for any negative element $\langle C, D\rangle$ of $\overline{R}$ we have $\inf\{x \cdot y^{-1} : x \in C, y \in D \cap (-\infty, 0)\} = 1$. For any elements $\langle C_1, D_1\rangle$, $\langle C_2, D_2\rangle$ of $\overline{R}$, we define

$$\langle C_1, D_1\rangle + \langle C_2, D_2\rangle := \langle R \setminus (D_1 + D_2), D_1 + D_2\rangle,$$

$$\langle C_1, D_1\rangle \cdot \langle C_2, D_2\rangle$$

$$:= \begin{cases} \langle R \setminus (D_1 \cdot D_2), D_1 \cdot D_2\rangle & \text{if } 0 \in C_1 \text{ and } 0 \in C_2 \\ \langle \mathrm{cl}(-((-C_1) \cdot D_2)), R \setminus \mathrm{cl}(-((-C_1) \cdot D_2))\rangle & \text{if } 0 \in D_1 \text{ and } 0 \in C_2 \\ \langle \mathrm{cl}(-(D_1 \cdot (-C_2))), R \setminus \mathrm{cl}(-(D_1 \cdot (-C_2)))\rangle & \text{if } 0 \in C_1 \text{ and } 0 \in D_2 \\ \langle R \setminus \mathrm{int}(C_1 \cdot C_2), \mathrm{int}(C_1 \cdot C_2)\rangle & \text{if } 0 \in D_1 \text{ and } 0 \in D_2, \end{cases}$$

where for each $n \in \mathbb{N}$ the topological closure in R^n of a set $A \subseteq R^n$ is denoted by $\mathrm{cl}(A)$, and its topological interior in R^n by $\mathrm{int}(A)$. Then, by the argument of [6], the structure $(\overline{R}, <, +, \cdot)$ is an ordered field and the structure $(R, <, +, \cdot)$ is a subfield of it.

Recall the notion of the strong monotonicity from [6].

Definition 2.1. We say that a weakly o-minimal structure $\mathcal{R} = (R, <, \ldots)$ has *the strong monotonicity* if for each definable set $I \subseteq R$ and each definable function $f : I \to \overline{R}$, there exists a partition of I into a finite set X and definable convex open sets $I_1, \ldots, I_k$ such that for each $i \in \{1, \ldots, k\}$, one of the following conditions holds.

(1) $f|I_i$ is constant.
(2) $f|I_i$ is strictly increasing and for any $a, b \in I_i$ with $a < b$ and any $c, d \in R$ with $f(a) < c < d < f(b)$, there exists some $x \in (a, b)$ such that $c < f(x) < d$; in particular $f|I_i$ is continuous.
(3) $f|I_i$ is strictly decreasing and for any $a, b \in I_i$ with $a < b$ and any $c, d \in R$ with $f(a) > c > d > f(b)$, there exists some $x \in (a, b)$ such that $c > f(x) > d$; in particular $f|I_i$ is continuous.

Theorem 2.2 ([6, Lemma 1.4]). *Suppose that $\mathcal{R} = (R, <, +, \ldots)$ is a weakly o-minimal expansion of an ordered group $(R, <, +)$. Then the following conditions are equivalent.*

(1) *$\mathcal{R}$ is non-valuational.*
(2) *$\mathcal{R}$ has the strong monotonicity.*

3. Completions in weakly o-minimal structures

In this section, we calculate the completion of some weakly o-minimal structures.

Let $\mathcal{M}_1 = (\mathbb{Q}, <, P_1)$, where the unary relation symbol P_1 is interpreted by the convex set $(-\infty, \sqrt{2}) \cap \mathbb{Q}$. Then, by [1], the structure $\mathcal{M}_1$ is weakly o-minimal.

Proposition 3.1. *We have that* $\overline{\mathcal{M}}_1 = \mathbb{Q} \cup \{\sqrt{2}\}$*. Here we consider the cut* $\langle(-\infty, \sqrt{2}), (\sqrt{2}, +\infty)\rangle$ *the point* $\sqrt{2}$*.*

Proof. Because the structure $\mathcal{M}_1$ admits elimination of quantifiers, we have the proposition. □

Let $\mathcal{M}_2 = (\mathbb{Q}, <, P_2)$, where the unary relation symbol P_2 is interpreted by the convex set $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$. Then, by [1], the structure $\mathcal{M}_2$ is weakly o-minimal.

Proposition 3.2. *We have that* $\overline{\mathcal{M}}_2 = \mathbb{Q} \cup \{\sqrt{2}, -\sqrt{2}\}$*. Here we consider the cuts* $\langle(-\infty, \sqrt{2}), (\sqrt{2}, +\infty)\rangle, \langle(-\infty, -\sqrt{2})$ *and* $(-\sqrt{2}, +\infty)\rangle$ *the points* $\sqrt{2}$ *and* $-\sqrt{2}$*, respectively.*

Proof. Let $\mathcal{M}'_2 := (\mathcal{M}_2, P'_2)$, where the unary relation symbol P'_2 is interpreted by the convex set $(-\infty, -\sqrt{2}) \cap \mathbb{Q}$. Then, the structure $\mathcal{M}'_2$ admits elimination of quantifiers. It follows that $\overline{\mathcal{M}}_2 = \mathbb{Q} \cup \{\sqrt{2}, -\sqrt{2}\}$. □

Let $\mathcal{M}_3 = (\mathbb{Q}, <, 0, +, -, P_3)$, where the unary relation symbol P_3 is interpreted by the convex set $(-\infty, \sqrt{2}) \cap \mathbb{Q}$. Then, by [1], the structure $\mathcal{M}_3$ is weakly o-minimal.

Proposition 3.3. *We have that* $\overline{\mathcal{M}}_3 = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$*. Here we consider the cut* $\langle(-\infty, a + b\sqrt{2}), (a + b\sqrt{2}, +\infty)\rangle$ *the point* $a + b\sqrt{2}$ *for each* $a, b \in \mathbb{Q}$*.*

Proof. It was proved in [4] that its definable expansion

$$\mathcal{M}'_3 := (\mathbb{Q}, <, 0, +, -, P_3, f_m(x))_{1<m<\omega}$$

admits elimination of quantifiers; here $f_m(x) = x/m$. It follows that $\overline{\mathcal{M}}_3 = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. □

4. Differentiability of definable functions

Throughout this section, we assume that $\mathcal{R} = (R, <, +, \cdot, \ldots)$ is a non-valuational weakly o-minimal expansion of a real closed field $(R, <, +, \cdot)$.

Lemma 4.1 ([6, Lemma 1.2])**.** *Let* $I \subseteq R$ *be a non-empty definable convex open set and* $f : I \to \overline{R}$ *a definable function. Then the limits* $\lim_{x\to\sup I-0} f(x)$ *and* $\lim_{x\to\inf I+0} f(x)$ *exist in* $\overline{R} \cup \{-\infty, +\infty\}$*.*

Let I be a non-empty definable open subset of R and $f : I \to \overline{R}$ a definable function. For each $x \in I$, we define the limits

$$f'_+(x) := \lim_{t\to+0} \frac{f(x+t)-f(x)}{t},$$
$$f'_-(x) := \lim_{t\to-0} \frac{f(x+t)-f(x)}{t},$$
$$f'(x) := \lim_{t\to 0} \frac{f(x+t)-f(x)}{t}.$$

Lemma 4.2 ([5, Lemma 3.3]). *Let $I \subseteq R$ be a non-empty definable convex open set and $f : I \to \overline{R}$ a definable function. Then, for each $x \in I$ the limits $f'_+(x)$ and $f'_-(x)$ exist in $\overline{R} \cup \{-\infty, +\infty\}$.*

Proof. Suppose that $x \in I$. We define $g(t) := t^{-1}(f(x+t) - f(x))$ on an open interval $(0, \varepsilon)$. Then, by Lemma 4.1, the limit $f'_+(x) = \lim_{t\to+0} g(t)$ exists in $\overline{R} \cup \{-\infty, +\infty\}$. Similarly, the limit $f'_-(x)$ exists in $\overline{R} \cup \{-\infty, +\infty\}$. □

Lemma 4.3. *Let I be a non-empty definable convex open subset of R. Let $f : I \to \overline{R}$ be a definable function. Then, the sets $\{x \in I : f'_+(x) = +\infty\}$ and $\{x \in I : f'_+(x) = -\infty\}$ are definable.*

Proof. We only show that the set $\{x \in I : f'_+(x) = +\infty\}$ is definable. Let $A := \{x \in I : f'_+(x) = +\infty\}$. Suppose that $x \in I$. Then, there exists $\varepsilon > 0$ such that $(x, x+\varepsilon) \subseteq I$. We define $g_{x,\varepsilon}(t) := t^{-1}(f(x+t)-f(x))$ on open interval $(0, \varepsilon)$. Then, we have $A = \{x \in I :$ there exists $\varepsilon > 0$ such that $(x, x+\varepsilon) \subseteq I$ and $\lim_{t\to+0} g_{x,\varepsilon}(t) = +\infty\}$. By weak o-minimality of $\mathcal{R}$, it follows that $\lim_{t\to+0} g_{x,\varepsilon}(t) = +\infty$ if and only if for all $c \in R$ there exists $\delta \in R$ with $0 < \delta < \varepsilon$ such that $g_{x,\varepsilon}(t) > c$ for all $t \in (0, \delta)$. Also, $g_{x,\varepsilon}(t) > c$ for all $t \in (0, \delta)$ if and only if for all $t \in (0, \delta)$ there exists $d > c$ such that $\langle t, d\rangle \in \Gamma_<(g_{x,\varepsilon})$. It follows that A is definable. □

By Lemmas 4.2 and 4.3, we have the following.

Lemma 4.4. *Let I be a non-empty definable convex open subset of R. Let $f : I \to \overline{R}$ be a definable function. Then, the set $\{x \in I : f'_+(x)$ exists in $\overline{R}\}$ is definable.*

Theorem 4.5. *Let I be a non-empty definable convex open subset of R. Let $f : I \to \overline{R}$ be a definable function. Then the set $\{x \in I : f'(x)$ exists in $\overline{R}\}$ is definable.*

Proof. Let $A := \{x \in I : f'_+(x), f'_-(x)$ exist in $\overline{R}\}$. By Lemma 4.4, the set A is definable. For each $x \in A$, we define $g_{x,\varepsilon}(t) := t^{-1}(f(x+t) -$

$f(x))$ on some set $(-\varepsilon, \varepsilon) \setminus \{0\}$ with $(x-\varepsilon, x+\varepsilon) \subseteq I$. We fix $x \in A$. By Theorem 2.2, there exists $\varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subseteq I$ such that $g_{x,\varepsilon}|(0,\varepsilon)$ and $g_{x,\varepsilon}|(-\varepsilon, 0)$ are strictly monotone or constant.

Without loss of generality, we may assume that both $g_{x,\varepsilon}|(0,\varepsilon)$ and $g_{x,\varepsilon}|(-\varepsilon,0)$ are strictly increasing. Now, $g_{x,\varepsilon}|(0,\varepsilon)$ is strictly increasing if and only if for each $t_1, t_2 \in (0,\varepsilon)$ with $t_1 < t_2$ there exists $y \in R$ such that $\langle t_1, y\rangle \notin \Gamma_<(g_{x,\varepsilon})$ and $\langle t_2, y\rangle \in \Gamma_<(g_{x,\varepsilon})$. Moreover we have that $\lim_{t\to -0} g_{x,\varepsilon}(t) < \lim_{t\to +0} g_{x,\varepsilon}(t)$ if and only if there exist $a, b \in R$ with $a < b$ such that for each $s \in (-\varepsilon, 0)$ and for each $y \in R$, if $\langle s, y\rangle \in \Gamma_<(g_{x,\varepsilon})$ then $y < a$, and for each $t \in (0,\varepsilon)$ there exists $z \in R$ with $b < z$ such that $\langle t, z\rangle \in \Gamma_<(g_{x,\varepsilon})$. We also have that $\lim_{t\to +0} g_{x,\varepsilon}(t) < \lim_{t\to -0} g_{x,\varepsilon}(t)$ if and only if there exist a, s, t, y, z with $s \in (-\varepsilon, 0)$ and $t \in (0, \varepsilon)$ such that $z < a < y$, $\langle s, y\rangle \in \Gamma_<(g_{x,\varepsilon})$ and $\langle t, z\rangle \in \Gamma_<(g_{x,\varepsilon})$. It follows that the set $\{x \in I : f'(x) \text{ exists in } \overline{R}\}$ is definable. □

Theorem 4.6. *Let I be a non-empty definable convex open subset of R. Let $f : I \to \overline{R}$ be a definable function. Suppose that the function f is differentiable at all points of I. Then, the function $f' : I \to \overline{R}$ is definable.*

Proof. Let $\Gamma_\le(f'_+) := \{\langle x, y\rangle \in I \times R : y \le f'_+(x)\}$. For each $x, y \in R$, we have that $\langle x, y\rangle \in \Gamma_<(f'_+)$ if and only if there exists z such that $y < z$ and $\langle x, z\rangle \in \Gamma_\le(f'_+)$. Hence, it suffices to show that $\Gamma_\le(f'_+)$ is definable.

We fix $x \in I$. By Theorem 2.2, there exists $\varepsilon > 0$ with $(x, x+\varepsilon) \subseteq I$ such that the definable function $g : (0,\varepsilon) \to \overline{R}$ defined by $g(t) := t^{-1}(f(x+t) - f(x))$ is strictly monotone or constant. We assume that the function g is strictly increasing on the open interval $(0,\varepsilon)$. The other cases are similar. Then, for each $y \in R$ we obtain that $\langle x, y\rangle \in \Gamma_\le(f'_+)$ if and only if $\langle t, y\rangle \in \Gamma_<(g)$ for each $t \in (0,\varepsilon)$. It follows that $\Gamma_\le(f'_+)$ is definable. □

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Department of Mathematics, Faculty of Science, Okayama University 1-1, Naka 3-chome, Tsushima, Okayama 700-8530, Japan
E-mail address: htanaka@math.okayama-u.ac.jp

Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan
E-mail address: kawa@center.wakayama-u.ac.jp